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# ON LORENTZ SPACES $\Gamma_{p,w}$

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ABSTRACT

We study Lorentz spaces $\Gamma_{p,w}$, where $0 < p < \infty$, and $w$ is a nonnegative measurable weight function. We first present some results concerning new formulas for the quasi-norm, duality, embeddings and Boyd indices. We then show that, whenever $\Gamma_{p,w}$ does not coincide with $L^1 + L^\infty$, it contains an order isomorphic and complemented copy of $\ell^p$. We apply this result to determine criteria for order convexity and concavity as well as for lower and upper estimates. Finally, we characterize the type and cotype of $\Gamma_{p,w}$.

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## 0. Preliminaries

In this paper we study the Lorentz spaces $\Gamma_{p,w}$ for arbitrary $0 < p < \infty$ and any measurable weight function $w \geq 0$. These spaces arise naturally in interpolation theory as a result of the Lions–Peetre $K$-method, and are naturally related to classical Lorentz spaces $\Lambda_{p,w}$. For arbitrary $0 < p < \infty$, the spaces $\Lambda_{p,w}$ and $\Gamma_{p,w}$ coincide if and only if the Hardy operator $H^1 f = f^{**}$ is bounded on $\Lambda_{p,w}$, which in turn is equivalent to the condition that the weight function $w$ satisfies the so-called $B_p$ condition ([3, 35, 33, 2, 34, 26, 36, 8]). In the case when $w(x) = x^{p/q-1}, 1 < p, q < \infty$, this was already observed long ago by Hunt in [17]. A result of Sawyer (Theorem 1 in [35]) emphasizes this relationship between $\Lambda$ and $\Gamma$ spaces even more. In fact, Sawyer proved that the Köthe dual of $\Lambda_{p,w}$, for $1 < p < \infty$ and under the assumption that $\int_0^\infty w = \infty$, coincides with the space $\Gamma_{p',\widetilde{w}}$, where $1/p + 1/p' = 1$ and $\widetilde{w}(x) = (x/\int_0^x w)^{p'} w(x)$.

This paper is divided into four sections. The introductory section contains all necessary notations, definitions and auxiliary results which will be needed later. Here we also introduce a new condition on weight functions $w$, called **reverse** $B_p$ and denoted by $RB_p$, which plays a crucial role in further investigations, especially in studying duality relations between the spaces $\Lambda_{p,w}$ and $\Gamma_{p',\widetilde{w}}$.

In section 1 we establish some basic properties of $\Gamma_{p,w}$. We start with new formulas for the quasi-norm in $\Gamma_{p,w}$ and a list of several simple but very useful properties of the fundamental function $\varphi$ of $\Gamma_{p,w}$. Then we present results about inclusions between $\Gamma_{p,w}$ and $\Gamma_{q,v}$, as well as between $\Gamma_{p,w}$ and $L^1 + L^\infty$ or $L^1 \cap L^\infty$. As a consequence we observe that $\Gamma_{p,w}$ always has a non-trivial dual. We also characterize when the dual of $\Lambda_{p,w}$ is non-trivial. We show further that, for $1 < p < \infty$, conditions $B_p$ and $RB_{p'}$ are dual to each other in the sense that $w$ satisfies $B_p$ if and only if $\widetilde{w}$ satisfies $RB_{p'}$, where $1/p + 1/p' = 1$. We then apply this fact and Sawyer's characterization of the dual space of $\Lambda_{p,w}$ to provide descriptions of dual and predual spaces of $\Gamma_{p,w}$, under the assumption that $w$ satisfies condition $RB_p$. We also establish a connection between regularity of the fundamental function $\varphi$ of $\Gamma_{p,w}$ and condition $RB_p$. In particular we show that for $0 < p \leq 1$, $w$ satisfies condition $RB_p$ if and only if $\varphi$ is regular. Finally, we find formulas for the Boyd indices of $\Gamma_{p,w}$ and show that $\Gamma_{p,w}$ contains copies of $\ell_n^\infty$ uniformly whenever its fundamental function fails to be regular.

In section 2 we prove that for any $0 < p < \infty$, if $\Gamma_{p,w}$ does not coincide with $L^1 + L^\infty$, then it contains an order almost isometric and complemented copy of $\ell^p$. For $1 < p < \infty$ and some extra assumptions on $w$, this is also a consequence of Levy's theorem, that an interpolation space obtained by the real $K$-method contains a copy of $\ell^p$ [27].

In section 3 we provide characterizations of order convexity and concavity as well as lower and upper estimates in $\Gamma_{p,w}$. The cases $0 < p \leq 1$ and $1 < p < \infty$ are studied separately. In view of the description of a predual space in section 1, the characterizations for $1 < p < \infty$ are obtained by duality from the appropriate results for Lorentz spaces $\Lambda_{p,w}$, thoroughly investigated in [21, 22]. For $0 < p \leq 1$ we use different methods, obtaining the estimations by direct calculations. At the end of the section we characterize the type and cotype of $\Gamma_{p,w}$ for $1 \leq p < \infty$.

We start with some notions and definitions which we will need later in the paper. In the following $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+ = [0, \infty)$ stand for the sets of natural, real and nonnegative real numbers, respectively.

A quasi-Banach lattice $X = (X, \|\cdot\|)$ is said to be **$p$-convex**, $0 < p < \infty$, respectively **$p$-concave**, $0 < p < \infty$, if there is a constant $C > 0$ such that

$$\left\| \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \leq C \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}$$

respectively,

$$\left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \leq C \left\| \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\|$$

for every choice of vectors $x_1, \ldots, x_n \in X$. We also say that $X$ satisfies an **upper $p$-estimate**, $0 < p < \infty$, respectively a **lower $p$-estimate**, $0 < p < \infty$, if the definition of $p$ -convexity, respectively $p$-concavity, holds true for any choice of disjointly supported elements $x_1, \ldots, x_n$ in $X$ ([18, 28]).

Recall that given $0 < p < \infty$, if $X$ is $p$-convex (resp. $p$-concave), then $X$ is $r$-convex (resp. $r$-concave) for $0 < r < p$ (resp. $r > p$). Lower and upper estimates are related in a similar way ([9, 18, 23]). Recall also that $(X^{(p)}, \|\cdot\|_{X^{(p)}})$, where $X^{(p)} = \{x : |x|^p \in X\}$ and $\|x\|_{X^{(p)}} = \||x|^p\|^{1/p}$, is a quasi-Banach lattice called the $p$-convexification of $X$. Clearly $X$ is 1-convex, that is $X$ is a Banach lattice, if and only if $X^{(p)}$ is $p$-convex.

We say that a quasi-Banach lattice $(X, \|\cdot\|)$ is **normable** whenever there exists a norm $|||\cdot|||$ in $X$ such that $C^{-1}\|x\| \leq |||x||| \leq C\|x\|$ for all $x \in X$ and some $C > 0$. We can always assume that $|||\cdot|||$ is a lattice norm on $X$ with respect to the same order. In fact it is enough to take

$$|||x||| = \inf \left\{ \sum_{i=1}^{n} \|x_i\| : x = \sum_{i=1}^{n} x_i, x_i \in X, n \in \mathbb{N} \right\}.$$

By $L^0$ we denote the space of all (equivalence classes of) Lebesgue-measurable real valued functions $f$ on $\mathbb{R}_+$. A quasi-normed lattice $(E, \|\cdot\|_E)$ is called a

**quasi-normed function space** if it is a sublattice of $L^0$ satisfying the following conditions:

(1) If $f \in L^0, g \in E$ and $|f| \leq |g|$ a.e., then $f \in E$ and $\|f\|_E \leq \|g\|_E$.

(2) There exists a strictly positive $f \in E$.

If $E = (E, \|\cdot\|_E)$ is complete then it is called a **quasi-Banach function space**. We say that the quasi-norm $\|\cdot\|_E$ or the space $(E, \|\cdot\|_E)$ is **order continuous** if for any $f \in E$ and $|f_n| \leq |f|$ with $|f_n| \to 0$, $\|f_n\|_E \to 0$. Recall also that $(E, \|\cdot\|_E)$ has the **Fatou property**, if whenever $0 \leq f_n \in E$ for $n \in \mathbb{N}$, $f \in L^0, f_n \uparrow f$ a.e. and $\sup_n \|f_n\|_E < \infty$, then $f \in E$ and $\|f_n\|_E \uparrow \|f\|_E$.

As usual, $E^*$ and $E'$ will stand for the topological and Köthe duals of $E$, respectively. Recall that the Köthe dual $E'$ is defined as

$$E' = \{f \in L^0 : \|f\|_{E'} = \sup_{\|g\|_E \leq 1} \int |fg| < \infty\},$$

and that $(E', \|\cdot\|_{E'})$ is a Banach lattice. Below we will use the well known fact that quasi-norm in $E$ is order continuous if and only if $E$ does not contain an isomorphic copy of $\ell^\infty$. This in turn is equivalent to the separability of $E$. Moreover, if one of these conditions is satisfied then $E^*$ is lattice isometric to $E'$, which will be denoted further by $E^* \cong E'$ (cf. [1, 25]).

A quasi-Banach function space $E$ on $\mathbb{R}_+$ is said to be **rearrangement invariant** (or **r.i.**) if for every $f \in L^0$ and $g \in E$ with $d_f = d_g$, we have $f \in E$ and $\|f\|_E = \|g\|_E$. Recall that $d_f$ denotes the distribution function of $f$, i.e., $d_f(\lambda) = |\{t \in \mathbb{R}_+ : |f(t)| > \lambda\}|$, $\lambda \geq 0$, where $||$ is the Lebesgue measure on $\mathbb{R}$. Then the **nonincreasing rearrangement** $f^*$ of $f$ is defined by $f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) \leq t\}, t \in \mathbb{R}_+$. Given a r.i. quasi-Banach function space $E$, let $\varphi_E$ denote its fundamental function, that is $\varphi_E(0) = 0$ and $\varphi_E(t) = \|\chi_{(0,t)}\|_E, t > 0$. The **lower** and **upper Boyd** indices of $E$ are defined as follows:

$$p(E) = \sup\{p > 0 : \text{there exists } C > 0,\ \|D_a\| \leq Ca^{-1/p} \text{ for all } 0 < a < 1\},$$
$$q(E) = \inf\{q > 0 : \text{there exists } C > 0,\ \|D_a\| \leq Ca^{-1/q} \text{ for all } a > 1\},$$

where $D_a$: $E \to E$, $a > 0$, is the dilation operator defined by $D_a f(t) = f(at)$, $t \in \mathbb{R}_+$ ([16, 25, 28]).

Throughout the paper the terms decreasing or increasing will always mean nonincreasing or nondecreasing, respectively.

A function $F$: $\mathbb{R}_+ \to \mathbb{R}_+$ is said to be **pseudo-increasing** (resp. **pseudo-decreasing**) whenever there exists $C > 0$ such that $F(u) \leq CF(v)$ (resp. $F(u) \geq CF(v)$) for all $0 \leq u < v$. The notation $A \approx B$ indicates that the expressions

$A$ and $B$ are equivalent, that is $A/B$ is bounded above and below by positive constants. Given a function $F\colon \mathbb{R}_+ \to \mathbb{R}_+$, we define the **lower** and **upper Matuszewska–Orlicz indices** ([25, 30]) as follows:

$$\alpha(F) = \sup\{p \in \mathbb{R}\colon F(au) \le Ca^p F(u) \text{ for some } C>0 \text{ and all } u \in \mathbb{R}_+, 0<a\le 1\},$$
$$\beta(F) = \inf\{p \in \mathbb{R}\colon F(au) \le Ca^p F(u) \text{ for some } C>0 \text{ and all } u \in \mathbb{R}_+, a \ge 1\}.$$

If $F \approx G$ then their corresponding indices coincide. It is well known that $\alpha(u^p \cdot F(u)) = p + \alpha(F)$ for any $p \in \mathbb{R}$, and $\alpha(F^a) = a\alpha(F)$ for $a > 0$ and $\alpha(F^a) = a\beta(F)$ for $a < 0$. The corresponding properties also hold for the upper index. An increasing function $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called **regular** whenever $\alpha(F) > 0$. For more information about indices we refer the reader to [24, 25, 30].

For each $r \in (0,\infty)$ we define operators $H^r$ and $H_r$ acting on $L^0$. These are special cases of the Hardy operators studied in [31] and are given by

$$H^r f(t) = \left(\frac{1}{t}\int_0^t f^{*r}(s)ds\right)^{1/r} \quad \text{and} \quad H_r f(t) = \left(\frac{1}{t}\int_t^\infty f^{*r}(s)ds\right)^{1/r}.$$

For $r = 1$, $H^1 f$ is usually denoted by $f^{**}$ and $H_1 f$ by $f_{**}$.

Any nonnegative function $w \in L^0$ is called a **weight function**. For $0 < p < \infty$, we say that a weight function $w$ belongs to the class $\mathcal{D}_p$ ($w \in \mathcal{D}_p$) whenever for all $x > 0$,

$$0 < W(x) := \int_0^x w(t)dt = \int_0^x w < \infty$$

and

$$W_p(x) := x^p \int_x^\infty t^{-p} w(t)dt < \infty.$$

Given $0 < p < \infty$ and $w \in \mathcal{D}_p$, the **Lorentz space** $\Gamma_{p,w}$ is then defined as the set of all $f \in L^0$ such that

$$\|f\| := \|f\|_{\Gamma_{p,w}} = \left(\int_0^\infty f^{**p}(t)w(t)dt\right)^{1/p} = \left(\int_0^\infty f^{**p} w\right)^{1/p} < \infty.$$

In view of the inequality $(f+g)^{**} \le f^{**} + g^{**}$, $f, g \in L^0$, it is standard to show that $\Gamma_{p,w}$ is a r.i. quasi-Banach function space with the Fatou property (cf. [4, 20, 25]). Observe also that the conditions imposed on $w$ are necessary and sufficient for $\Gamma_{p,w}$ to be a non-trivial quasi-normed function space. In fact, it is easy to show that $w \in \mathcal{D}_p$ if and only if there exists $0 < f \in \Gamma_{p,w}$ and $\|\cdot\|$ is a quasi-norm. The fundamental function of $\Gamma_{p,w}$ is given by the formula

$$\varphi(x) := \varphi_{\Gamma_{p,w}}(x) = (W(x) + W_p(x))^{1/p} = \left(px^p \int_x^\infty t^{-p-1} W(t)dt\right)^{1/p}, \; x > 0,$$

where the last equality follows from integration by parts. More explicitly, for any $\epsilon > 0$ there exists $x > 0$ such that $\int_x^\infty t^{-p}w(t)dt < \epsilon$. Hence, for sufficiently large $y > x$, $y^{-p}\int_0^y w \le y^{-p}\int_0^x w + \int_x^\infty t^{-p}w(t)dt < 2\epsilon$, and so $\lim_{y\to\infty} y^{-p}W(y) = 0$. Now, integrating by parts we obtain the desired equality. Let us also observe that $\varphi(x)/2 \le \psi(x) \le \varphi(x)$ for all $x > 0$, where

$$\psi(x) = \left(x^p \int_0^\infty \frac{w(t)}{(t+x)^p} dt\right)^{1/p}, \quad x > 0.$$

The function $\psi$ was already used in [13].

Let us also recall that, given $0 < p < \infty$ and a weight function $w$, the classical Lorentz space $\Lambda_{p,w}$ ([29]) consists of all $f \in L^0$ such that

$$\|f\|_{\Lambda_{p,w}} = \left(\int_0^\infty f^{*p}(t)w(t)dt\right)^{1/p} = \left(\int_0^\infty f^{*p}w\right)^{1/p} < \infty.$$

It is well known [14, 22] that $\|\cdot\|_{\Lambda_{p,w}}$ is a quasi-norm if and only if $W$ satisfies the $\Delta_2$-condition, that is

$$W(2t) \le KW(t)$$

for all $t > 0$ and some $K > 0$. Recently, it has been discovered that for certain choices of weights, $\Lambda_{p,w}$ may not be even a linear space. In fact, it was proved ([10], Corollary 1.5) that if $W$ is positive on $(0,\infty)$, then $\Lambda_{p,w}$ is linear if and only if $W$ satisfies the $\Delta_2$-condition.

It is also well known and not difficult to show that if $W$ satisfies the $\Delta_2$-condition and $W(\infty) := \int_0^\infty w = \infty$, then for any $0 < p < \infty$, $\Lambda_{p,w}$ is a separable r.i. quasi-Banach space. In fact, following the proof of part (1) of Proposition 1.4 below, $\|\cdot\|_{\Lambda_{p,w}}$ is order continuous if and only if $W(\infty) = \infty$.

We will also need the following spaces (cf. [4, 25]):

$$L^1 \cap L^\infty = \{f \in L^0 : \|f\|_{L^1\cap L^\infty} = \max(\|f\|_{L^1}, \|f\|_{L^\infty}) < \infty\},$$

$$L^1 + L^\infty = \{f \in L^0 : \|f\|_{L^1+L^\infty} = \int_0^1 f^*(s)ds < \infty\}.$$

Obviously, for any $0 < p < \infty$,

$$\Gamma_{p,w} \subset \Lambda_{p,w} \quad \text{and} \quad \Gamma_{p,w} \subset L^1 + L^\infty.$$

It is well known that $\Gamma_{p,w} = \Lambda_{p,w}$ if and only if $w$ satisfies **condition** $B_p$, that is there exists $A > 0$ such that for all $x > 0$,

$$\int_x^\infty t^{-p}w(t)dt \le Ax^{-p}\int_0^x w.$$

This important fact was proved by Ariño and Muckenhoupt [3] and Sawyer [35] for $1 < p < \infty$, and by Andersen [2], Lai [26] and Stepanov [36] (see also [8, 15]) for $0 < p \leq 1$.

The space $\Gamma_{p,w}$ has some interesting properties. In particular, for any $0 < p < \infty$, its dual is always non-trivial. Moreover, it is an interpolation space between $L^1$ and $L^\infty$, and in fact, $\Gamma_{p,w} = (L^1, L^\infty)_{\Phi,K}$, where

$$\Phi(f) = \left( \int_0^\infty t^{-p}|f(t)|^p w(t)dt \right)^{1/p},$$

and $K(t, f; L^1, L^\infty) = \int_0^t f^*(s)ds$ (cf. [5, 25]). Note that for $p = 1$, $\Gamma_{1,w}$ coincides with the classical Lorentz space $\Lambda_{1,v}$, where $v(x) = \varphi'(x) = \int_x^\infty s^{-1}w(s)ds$. Denoting by $\Gamma_{q,p}$ the space $\Gamma_{p,w}$ for $w(t) = t^{p/q-1}$ we have that for $0 < q \leq 1$, $\Gamma_{q,p}(0, \infty) = \{0\}$, since $\int_x^\infty t^{-p}t^{p/q-1}dt = \infty$ for all $x > 0$. However, $\Gamma_{q,p}(0, 1) \neq \{0\}$, and in particular, $\Gamma_{1,1}(0, 1) = L \log L$.

We now introduce a new condition on weights $w$, called $RB_p$. As we shall see, it plays an important role in many problems concerning the spaces $\Gamma_{p,w}$ and may be considered as a dual condition to $B_p$.

Given $0 < p < \infty$, we say that a weight function $w \in \mathcal{D}_p$ satisfies the **reverse** $B_p$ **condition**, denoted also by $RB_p$, if there exists $C > 0$ such that for all $x > 0$,

$$x^{-p} \int_0^x w \leq C \int_x^\infty t^{-p}w(t)dt.$$

It is clear that if $w$ satisfies condition $RB_p$ then $\varphi \approx W_p^{1/p}$.

In [35, Theorem 1] Sawyer characterized the Köthe dual of $\Lambda_{p,w}$, providing a comparatively simple formula equivalent to the norm $\|\cdot\|_{\Lambda'_{p,w}}$, by discovering a connection with the spaces $\Gamma_{p',\widetilde{w}}$. If, in addition, one assumes that $\Lambda_{p,w}$ is an order continuous quasi-Banach space, then Sawyer's result also provides a characterization of its topological dual $\Lambda^*_{p,w}$ with a norm equivalent to $\|\cdot\|_{\Lambda^*_{p,w}}$. In fact we have the following result.

THEOREM 0.1: *Let $1 < p < \infty$ and assume that $w \geq 0$ is a measurable function such that $W(x) := \int_0^x w < \infty$ for all $x > 0$, $W$ satisfies the $\Delta_2$-condition, and $W(\infty) = \infty$. Then for every $F \in \Lambda^*_{p,w}$ there exists $g \in L^0$ such that for all $f \in \Lambda_{p,w}$*

$$F(f) = \int_0^\infty fg,$$

*and $\|F\| = \|g\|_{\Lambda'_{p,w}} \approx \|g\|_{\Gamma_{p',\widetilde{w}}}$ where $\widetilde{w}(x) = (x/W(x))^{p'}w(x)$. Consequently,*

$$\Lambda'_{p,w} = \Gamma_{p',\widetilde{w}}$$

*with* $\|\cdot\|_{\Gamma_{p',\tilde{w}}} \approx \|\cdot\|_{\Lambda'_{p,w}}$, *and* $\Lambda^*_{p,w}$ *is lattice isomorphic to* $\Gamma_{p',\tilde{w}}$.

## 1. Properties of $\Gamma_{p,w}$

For the rest of the paper we will assume, unless otherwise stated, that $0 < p < \infty$ and that $w$ is a weight function belonging to the class $\mathcal{D}_p$. We start by giving equivalent formulas for the quasi-norm of $\Gamma_{p,w}$. We prove them using standard methods (cf. [6, 33]).

PROPOSITION 1.1: *For every* $f \in \Gamma_{p,w}$,

$$\|f\| = \left(\int_0^\infty W(d_{f^{**}}(t))d(t^p)\right)^{1/p}.$$

*If, in addition,* $\int_1^\infty w = \infty$, *then for every* $f \in \Gamma_{p,w}$

$$\|f\| = \left(\int_0^\infty t^{-p}W_p(t)d\left(\left(\int_0^t f^*(s)ds\right)^p\right)\right)^{1/p}.$$

*Proof:* For any $f \in \Gamma_{p,w}$, applying Fubini's theorem, we obtain

$$\begin{aligned}\int_0^\infty f^{**p}(t)w(t)dt &= \int_0^\infty \left(\int_0^{f^{**}(t)} d(s^p)\right)w(t)dt\\ &= \int_0^\infty \left(\int_0^\infty \chi_{\{t: f^{**}(t)>s\}}(s,t)d(s^p)\right)w(t)dt\\ &= \int_0^\infty \left(\int_0^\infty \chi_{\{t: f^{**}(t)>s\}}(s,t)w(t)dt\right)d(s^p)\\ &= \int_0^\infty W(d_{f^{**}}(s))d(s^p).\end{aligned}$$

Integrating by parts, since $W_p(t)f^{**}(t)^p$ tends to 0 as $t$ tends to 0 or to $\infty$ for every $f \in \Gamma_{p,w}$, we have

$$\begin{aligned}\int_0^\infty t^{-p}W_p(t)d\left(\left(\int_0^t f^*(s)ds\right)^p\right) &= \int_0^\infty \left(\int_t^\infty u^{-p}w(u)du\right)d\left(\left(\int_0^t f^*(s)ds\right)^p\right)\\ &= \int_0^\infty \left(\int_0^u f^*(s)ds\right)^p u^{-p}w(u)du\\ &= \int_0^\infty f^{**p}(u)w(u)du. \quad ∎\end{aligned}$$

PROPOSITION 1.2: *Let $\varphi$ be the fundamental function of $\Gamma_{p,w}$ and define*

$$a_\varphi = \inf_{x>0} \frac{x\varphi'(x)}{\varphi(x)} \quad \text{and} \quad b_\varphi = \sup_{x>0} \frac{x\varphi'(x)}{\varphi(x)}.$$

*Then $\varphi$, $a_\varphi$ and $b_\varphi$ have the following properties.*

(1) *The function $\varphi(x)/x$ is decreasing on $(0,\infty)$.*

(2) *If $0 < p \le 1$, then $\varphi$ is concave on $(0,\infty)$.*

(3) *We have*

$$0 \le a_\varphi \le \alpha(\varphi) \le \beta(\varphi) \le b_\varphi \le 1.$$

*Moreover, $w$ satisfies condition $B_p$ (resp. $RB_p$) if and only if $b_\varphi < 1$ (resp. $a_\varphi > 0$).*

(4) *If $\alpha(\varphi) > 0$ then $W(x)/x$ satisfies condition $RB_p$.*

*Proof:* Since

$$\left(\frac{\varphi(x)}{x}\right)^p = p\int_x^\infty t^{-p-1}W(t)dt,$$

$\varphi(x)/x$ is decreasing. Moreover,

$$\varphi'(x) = \left(\frac{\varphi(x)}{x}\right)^{1-p} \int_x^\infty t^{-p}w(t)dt,$$

which shows that $\varphi'$ is decreasing when $0 < p \le 1$, and thus $\varphi$ is concave. Condition (3) is an immediate result of the equality

$$\frac{x\varphi'(x)}{\varphi(x)} = \frac{W_p(x)}{W(x)+W_p(x)}.$$

In order to prove (4), we observe that the assumption $\alpha(\varphi) > 0$ implies that $\varphi^p(x)/x^\epsilon$ is pseudo-increasing for some $\epsilon > 0$, and thus for some $C > 0$,

$$\frac{\widetilde{W}_p(x)}{x^\epsilon} \le C\frac{\widetilde{W}_p(y)}{y^\epsilon}, \quad \text{where } \widetilde{W}_p(x) = x^p\int_x^\infty t^{-p-1}W(t)dt.$$

Hence

$$\int_0^x \widetilde{W}_p(s)\frac{ds}{s} = \int_s^x \widetilde{W}_p(s)s^{-\epsilon}s^{\epsilon-1}ds \le \frac{C}{\epsilon}\widetilde{W}_p(x).$$

By changing the order of integration we obtain

$$\begin{aligned}\int_0^x \widetilde{W}_p(s)\frac{ds}{s} &= \int_0^x \left(\int_s^\infty t^{-p-1}W(t)dt\right)s^{p-1}ds \\ &= \int_0^x \left(\int_0^t s^{p-1}ds\right)t^{-p-1}W(t)dt + \int_x^\infty \left(\int_0^x s^{p-1}ds\right)t^{-p-1}W(t)dt \\ &= \frac{1}{p}\int_0^x \frac{W(t)}{t}dt + \frac{1}{p}x^p\int_x^\infty t^{-p-1}W(t)dt.\end{aligned}$$

Thus

$$\int_0^x \frac{W(t)}{t}dt \le ((C/p)\epsilon - 1)x^p \int_x^\infty t^{-p-1}W(t)dt,$$

which means that $W(x)/x$ satisfies condition $RB_p$. ∎

The second formula in Proposition 1.1 immediately yields the following embedding result. We wish to point out here that conditions (1) and (2) in the statement below are equivalent also when condition $RB_p$ does not hold (cf. [7, 15, 36]).

PROPOSITION 1.3: *Let $0 < p < \infty$ and suppose that both of the weight functions $w, v$ satisfy condition $RB_p$. Moreover, let $V(x) = \int_0^x v$ and $V_p(x) = x^{-p}\int_x^\infty t^{-p}v(t)dt, x > 0$. Then the following conditions are equivalent.*

(1) $\Gamma_{p,w} = \Gamma_{p,v}$ *and* $\|\cdot\|_{\Gamma_{p,w}} \approx \|\cdot\|_{\Gamma_{p,v}}$.

(2) $\varphi_{\Gamma_{p,w}} \approx \varphi_{\Gamma_{p,v}}$.

(3) $W_p \approx V_p$.

PROPOSITION 1.4: *The following conditions are satisfied.*

(1) $\Gamma_{p,w}$ *has order continuous norm if and only if* $\int_1^\infty w = \infty$.

(2) $L^1 \cap L^\infty \subset \Gamma_{p,w} \subset L^1 + L^\infty$.

(3) $L^\infty \subset \Gamma_{p,w}$ *if and only if* $\int_0^\infty w < \infty$.

(4) *The following equality holds true:*

$$\left(\int_0^\infty t^{-p}w(t)dt\right)^{1/p} = \lim_{x\to 0+}\frac{\varphi(x)}{x}.$$

*Consequently,* $L^1 \subset \Gamma_{p,w}$ *if and only if* $\int_0^\infty t^{-p}w(t)dt < \infty$.

(5) $\Gamma_{p,w} = L^1 + L^\infty$ *if and only if* $\int_0^\infty t^{-p}w(t)dt < \infty$ *and* $\int_0^\infty w < \infty$.

*Proof:* (1) If $\int_1^\infty w = \int_0^\infty w = \infty$ then $d_f(\lambda) < \infty$ for every $\lambda > 0$ and $f \in \Gamma_{p,w}$. Therefore any sequence $\{f_n\}$ satisfying $0 \le f_n \downarrow 0$ a.e. and $f_n \le f$ also satisfies $f_n^{**} \downarrow 0$. Thus $\|f_n\| \downarrow 0$, and so $\Gamma_{p,w}$ is order continuous. Assuming now that $\int_0^\infty w < \infty$ and taking $f_n = \chi_{(n,\infty)}$, we have that $f_n \downarrow 0$ and $\|f_n\| = \text{const.}$, which means that $\Gamma_{p,w}$ is not order continuous.

(2) For any $f \in \Gamma_{p,w}$,

$$\|f\|^p \ge \int_0^1 f^{**p}(1)w(t)dt + \int_1^\infty \left(\frac{1}{t}\int_0^1 f^*(s)ds\right)^p w(t)dt$$
$$= f^{**p}(1)\left(\int_0^1 w(t)dt + \int_1^\infty t^{-p}w(t)dt\right) = \varphi^p(1)\|f\|^p_{L^1+L^\infty},$$

and for $f \in L^1 \cap L^\infty$,

$$\|f\| \le \left( \int_0^1 \|f\|_{L^\infty}^p w(t)dt + \int_1^\infty ((1/t)\|f\|_{L^1})^p w(t)dt \right)^{1/p}$$
$$\le \max(\|f\|_{L^1}, \|f\|_{L^\infty}) \left( \int_0^1 w + \int_1^\infty t^{-p} w(t)dt \right)^{1/p} = \varphi(1)\|f\|_{L^1 \cap L^\infty}.$$

(3) It is clear that $L^\infty \subset \Gamma_{p,w}$ if and only if $f(x) \equiv 1$ belongs to $\Gamma_{p,w}$; thus $\int_0^\infty w < \infty$.

(4) Since the function $\varphi(x)/x$ is decreasing, its limit exists. If $\int_0^\infty t^{-p} w(t)dt < \infty$ then $\lim_{x\to 0+} W(x)/x^p = 0$, since $W(x)/x^p \le \int_0^x t^{-p} w(t)dt$. Consequently,

$$\lim_{x\to 0+} \frac{\varphi(x)}{x} = \lim_{x\to 0+} \left( \frac{W(x) + W_p(x)}{x^p} \right)^{1/p} = \left( \int_0^\infty t^{-p} w(t)dt \right)^{1/p}.$$

If $\int_0^\infty t^{-p} w(t)dt = \infty$ then

$$\lim_{x\to 0+} \frac{\varphi(x)}{x} \ge \lim_{x\to 0+} \left( \int_x^\infty t^{-p} w(t)dt \right)^{1/p} = \infty.$$

Now, if $L^1 \subset \Gamma_{p,w}$, then $\varphi(x) \le Kx$, and so $\lim_{x\to 0+} \varphi(x)/x < \infty$, which implies that $\int_0^\infty t^{-p} w(t)dt < \infty$. On the other hand, the latter inequality implies that for $f \in L^1$,

$$\|f\|^p = \int_0^\infty f^{**p} w \le \|f\|_{L^1}^p \int_0^\infty t^{-p} w(t)dt < \infty.$$

(5) This is an obvious consequence of (2), (3) and (4). ∎

The next result has been proved in [19] for $E$ over the interval $(0,1)$ and in [32] for $E$ over $(0,\infty)$. The proof we provide here relies on the idea from [19] and is simpler than the one in [32].

PROPOSITION 1.5: *Let $E$ be a separable r.i. quasi-Banach function space on $(0,\infty)$. Then $E^* \ne \{0\}$ if and only if $E \subset L^1 + L^\infty$.*

*Proof:* By the separability of $E$ we have that $E^* \cong E'$ (cf. [1, 25]). Assume that $E' \ne \{0\}$. Then there exists $F \in E'$ and $h \in L^0$, not identically equal to zero, such that $F(f) = \int_0^\infty fh$ for every $f \in E$. Without loss of generality we can assume that $h = h^*$. Let $0 < a \le 1$ be such that $h(a) > 0$. Then choosing a natural number $n$ such that $n \ge a^{-1}$, for any $f \in E$ we have

$$\|f\|_{L^1+L^\infty} = \int_0^1 f^* \le n \int_0^a f^* \le \frac{n}{h^*(a)} \int_0^a f^* h \le \frac{n}{h^*(a)} \|F\| \|f\|_E < \infty.$$

Thus $f \in L^1 + L^\infty$. ∎

Note that the assumption of separability in the above theorem is necessary. In fact, the space $L_{1,\infty}$, also called **Weak** $L^1$, of all measurable functions on $(0,\infty)$ such that

$$\|f\|_{1,\infty} = \sup_{t>0} t f^*(t) < \infty,$$

is a r.i. quasi-Banach function space, is non-separable and is not a subspace of $L^1 + L^\infty$, but has non-trivial dual [11, 14].

COROLLARY 1.6: *The following holds true.*

(1) *For any* $0 < p < \infty$, $\Gamma^*_{p,w} \neq \{0\}$.

(2) *Let* $w$ *be a weight function (not necessarily in the class* $\mathcal{D}_p$*) such that* $W$ *satisfies the* $\Delta_2$*-condition and* $W(\infty) = \infty$. *For* $0 < p \leq 1$, $\Lambda^*_{p,w} \neq \{0\}$ *if and only if*

$$\sup_{0<t<1} \frac{t}{W^{1/p}(t)} < \infty.$$

*For* $1 < p < \infty$, $\Lambda^*_{p,w} \neq \{0\}$ *if and only if*

$$\int_0^1 \left(\frac{t}{W(t)^{1/p}}\right)^{p'} \frac{dt}{t} = \int_0^1 \left(\frac{W(t)}{t}\right)^{1-p'} dt < \infty.$$

*Proof:* (1) Since $\Gamma_{p,w} \subset L^1 + L^\infty$, the dual of $\Gamma_{p,w}$ is not trivial.

(2) The assumption $W(\infty) = \infty$ ensures that $\Lambda_{p,w}$ is separable. Observe that $L^1 + L^\infty = \Lambda_{1,w_0}$, where $w_0(x) = 1$ for $0 \leq x \leq 1$ and zero otherwise. In view of Proposition 1 in [36] (see also Theorem 3.1 in [7]), if $0 < p \leq 1$, then $\Lambda_{p,w} \subset \Lambda_{1,w_0}$ if and only if

$$\sup_{t>0} \frac{\varphi_{\Lambda_{1,w_0}}(t)}{\varphi_{\Lambda_{p,w}}(t)} = \max\left\{ \sup_{0<t<1} \frac{t}{\varphi_{\Lambda_{p,w}}(t)}, \sup_{t>1} \frac{1}{\varphi_{\Lambda_{p,w}}(t)} \right\} = \sup_{0<t<1} \frac{t}{W^{1/p}(t)} < \infty,$$

and if $p > 1$, $\Lambda_{p,w} \subset \Lambda_{1,w_0}$ if and only if

$$\int_0^\infty \left(\frac{W_0}{W}\right)^{p'/p} w_0 = \int_0^1 \left(\frac{t}{W(t)}\right)^{1/(p-1)} dt = \int_0^1 \left(\frac{W(t)}{t}\right)^{1-p'} dt < \infty,$$

where $W_0(x) = \int_0^x w_0$.

The latter condition can also be obtained by Theorem 0.1. In fact, $\Lambda^*_{p,w} \neq \{0\}$ whenever $\Gamma_{p',\widetilde{w}} \neq \{0\}$, which is equivalent to

$$\int_0^x \left(\frac{t}{W(t)}\right)^{p'} w(t)dt + x^{p'} \int_x^\infty \frac{w(t)}{W^{p'}(t)} dt < \infty, \quad x > 0.$$

Integrating by substitution and by parts, the last formula is equal to

$$\frac{p'}{p'-1}\int_0^x \left(\frac{W(t)}{t}\right)^{1-p'} dt. \qquad \blacksquare$$

PROPOSITION 1.7: *Let $1 < p < \infty$ and let $w$ be a weight function such that $W(\infty) = \infty$. Then $w$ satisfies condition $B_p$ if and only if $\widetilde{w}$ satisfies condition $RB_{p'}$, where $\widetilde{w}(x) = (x/W(x))^{p'} w(x)$ and $1/p + 1/p' = 1$.*

*Proof:* Condition $RB_{p'}$ for $\widetilde{w}$ is equivalent to the following inequality:

$$x^{-p'}\int_0^x \left(\frac{t}{W(t)}\right)^{p'} w(t)dt \le C\int_x^\infty \frac{w(t)}{(W(t))^{p'}}dt, \quad x > 0.$$

In accordance with our definitions we can assume here that both sides of the inequality are finite. Integrating the left side by parts and making a substitution in the right side we obtain that, for all $x > 0$,

$$-\frac{1}{W^{(p'-1)}(x)} + x^{-p'}\lim_{t\to 0+}\frac{t^{p'}}{W^{(p'-1)}(t)} + p'x^{-p'}\int_0^x \left(\frac{t}{W(t)}\right)^{(p'-1)} dt \le \frac{C}{W^{(p'-1)}(x)}.$$

We next observe that $\lim_{x\to 0+} x^{p'}/W^{(p'-1)}(x) = 0$ in view of the inequality

$$\int_0^x \left(\frac{t}{W(t)}\right)^{p'-1} dt \ge \frac{x^{p'}}{p'W^{(p'-1)}(x)}.$$

Consequently, condition $RB_{p'}$ for $\widetilde{w}$ is equivalent to

$$u(x) \le \frac{C+1}{p'}\frac{1}{W^{(p'-1)}(x)} \quad \text{for all } x > 0,$$

where

$$u(x) = x^{-p'}\int_0^x \left(\frac{t}{W(t)}\right)^{p'-1} dt.$$

If $\widetilde{w}$ satisfies $RB_{p'}$ then for sufficiently small $\epsilon > 0$ and all $x > 0$ we have

$$\begin{aligned}\frac{d}{dx}\Big(\frac{u(x)}{x^{-p'+\epsilon}}\Big) &= \frac{d}{dx}\Big(x^{-\epsilon}\int_0^x \left(\frac{t}{W(t)}\right)^{p'-1} dt\Big)\\ &= x^{-\epsilon+p'-1}\Big(-\epsilon u(x) + \frac{1}{W^{(p'-1)}(x)}\Big)\\ &\ge \Big(-\epsilon\frac{C+1}{p'} + 1\Big)\frac{x^{-\epsilon+p'-1}}{W^{(p'-1)}(x)} > 0.\end{aligned}$$

Hence $u(x)/x^{-p'+\epsilon}$ is increasing and so $\alpha(u) \geq -p' + \epsilon > -p'$. But $\alpha(u) = -(p'-1)\beta(W)$, and it follows that $\beta(W) < p$ which is equivalent to condition $B_p$ for $w$ (cf. Theorem A in [22]).

Now if $w$ satisfies condition $B_p$, then $W(x)/x^{p-\epsilon}$ is pseudo-decreasing for some $\epsilon > 0$. Hence there exists $C > 0$ such that for all $x > 0$

$$u(x) = x^{-p'} \int_0^x \left(\frac{t^{p-\epsilon}}{W(t)}\right)^{p'-1} t^{(1-p+\epsilon)(p'-1)} dt$$
$$\leq Cx^{-p'} \left(\frac{x^{p-\epsilon}}{W(x)}\right)^{p'-1} x^{(p'-1)(1-p+\epsilon)+1} = \frac{C}{W^{(p'-1)}(x)},$$

which is equivalent to condition $RB_{p'}$ for $\widetilde{w}$. ∎

THEOREM 1.8: *Let $1 < p < \infty$ and let $w$ be a weight function satisfying condition $RB_p$ and also $\int_0^1 t^{-p}w(t)dt = \int_1^\infty w = \infty$. Let $v(x) = V'(x)$, where*

$$V(x) = \left(\int_x^\infty t^{-p}w(t)dt\right)^{-1/(p-1)}, \quad x > 0.$$

*Then $\Lambda_{p',v}$ is a normable space. Moreover, $\Lambda_{p',v}$ is a predual of $\Gamma_{p,w}$, that is*

$$\Gamma_{p,w} = \Lambda'_{p',v}$$

*with $\|\cdot\|_{\Gamma_{p,w}} \approx \|\cdot\|_{\Lambda'_{p',v}}$. Consequently, $\Gamma_{p,w}$ is lattice isomorphic to $\Lambda^*_{p',v}$.*

*Proof:* Since $w \in \mathcal{D}_p$, $V(\infty) = \infty$ and $V(x) < \infty$ for any $x > 0$. Moreover, $V(0) = 0$ in view of the assumption $\int_0^1 t^{-p}w(t)dt = \infty$. It is also clear that, for every $x > 0$,

$$v(x) = \frac{w(x)}{(p-1)x^p}\left(\int_x^\infty t^{-p}w(t)dt\right)^{-p/(p-1)},$$

and so $\widetilde{v}(x) = (x/V(x))^p v(x) = (1/(p-1))w(x)$. Hence $\widetilde{v}$ satisfies $RB_p$, and by Proposition 1.7, $v$ must satisfy condition $B_{p'}$. Therefore $\Lambda_{p',v}$ is normable (cf. [35]) and thus $V$ satisfies the $\Delta_2$-condition. We now apply Theorem 0.1, with $p'$ and $v$ in the roles of $p$ and $w$, to obtain that $\Lambda'_{p',v} = \Gamma_{p,\widetilde{v}} = \Gamma_{p,w}$ with $\| \|_{\Lambda'_{p',v}} \approx \| \|_{\Gamma_{p,w}}$. Finally, since $V(\infty) = \infty$, $\| \|_{p',v}$ is order continuous and so $\Lambda_{p',v}$ is separable. Consequently, its topological dual $\Lambda^*_{p',v}$ is lattice isomorphic to $\Gamma_{p,w}$. ∎

The next corollary provides a simple description of the dual space of $\Gamma_{p,w}$ under the assumption that $w$ satisfies condition $RB_p$ (cf. [13]). A referee pointed out to us that Theorem 2.7 in [12] is an analogous version of that result for spaces on the interval $(0, 1)$.

COROLLARY 1.9: *Let $1 < p < \infty$ and let $w$ be a weight function satisfying condition $RB_p$ and $\int_0^1 t^{-p}w(t)dt = \int_1^\infty w = \infty$. Then*

$$\Gamma^*_{p,w} \cong (\Gamma_{p,w})' = \Lambda_{p',v},$$

*where $v(x) = V'(x)$ and*

$$V(x) = \left(\int_x^\infty t^{-p}w(t)dt\right)^{-1/(p-1)}, \; x > 0.$$

*Proof:* Since, $\widetilde{v}(x) = (x/V(x)^p v(x) = Cw(x)$ for a.e. $x > 0$ and for some $C > 0$, $v$ satisfies condition $B_{p'}$ by Proposition 1.7. Hence $\Lambda_{p',v}$ is normable, and in fact $\Gamma_{p',v} = \Lambda_{p',v}$, and $\Gamma_{p',v}$ is a Banach function space with order continuous norm. $\Gamma_{p',v}$ also has the Fatou property and hence its second Köthe dual $(\Gamma_{p',v})''$ coincides with $\Gamma_{p',v}$. But $\Lambda^*_{p',v} = \Gamma^*_{p',v} = (\Gamma_{p',v})'$ and so

$$\Gamma^*_{p,w} = (\Gamma_{p,w})' = (\Lambda^*_{p',v})' = (\Gamma_{p',v})'' = \Gamma_{p',v} = \Lambda_{p',v}. \qquad \blacksquare$$

PROPOSITION 1.10: *Let $p \in (0,\infty)$ and let $w$ be a weight function in $\mathcal{D}_p$. Let $\varphi := \varphi_{\Gamma_{p,w}}$ be the fundamental function of $\Gamma_{p,w}$.*

*If $w$ satisfies condition $RB_p$ then $\varphi$ is regular, that is $\alpha(\varphi) > 0$.*

*Conversely, if $1 < p < \infty$ and $\alpha(\varphi) > 0$, then there exists a second weight function $w_0$ satisfying condition $RB_p$ such that the fundamental functions $\varphi_{\Gamma_{p,w}}$ and $\varphi_{\Gamma_{p,w_0}}$ are equivalent, that is $\Gamma_{p,w} = \Gamma_{p,w_0}$.*

*For $0 < p \leq 1$, $\alpha(\varphi) > 0$ if and only if $w$ satisfies $RB_p$.*

*Proof:* Suppose first that $w$ satisfies condition $RB_p$. For every $\epsilon > 0$ and almost every $x > 0$ we have

$$\frac{d}{dx}\left(\frac{\varphi^p(x)}{x^\epsilon}\right) = x^{-\epsilon-1}\left((p-\epsilon)x^p \int_x^\infty t^{-p}w(t)dt - \epsilon \int_0^x w\right).$$

If $\epsilon < p/(C+1)$, where $C$ is the constant in $RB_p$, then $(\varphi^p(x)/x^\epsilon)' > 0$ for almost all $x > 0$. Since $(\varphi^p(x)/x^\epsilon)$ is absolutely continuous on each compact subinterval of $(0,\infty)$, it follows that $\varphi^p(x)/x^\epsilon$ is increasing and so $\alpha(\varphi) > 0$.

Assume now that $1 < p < \infty$ and $\alpha(\varphi) > 0$. The second of these conditions ensures that the functions $\widehat{\psi}(x) := \int_0^x \varphi(t)/tdt$ and $\psi(x) := \int_0^x \widehat{\psi}(t)/tdt$ are both finite for all $x > 0$. Then, also using the fact (Proposition 1.2(1)) that $\varphi(x)/x$ is decreasing, we obtain that $\widehat{\psi}$, and so also $\psi$, is concave and equivalent to $\varphi$ (cf. [24]). Clearly $\psi$ is twice continuously differentiable.

Let us now define the function $w_0 \colon (0,\infty) \to \mathbb{R}$ by

$$w_0(x) = -x^p\Big(\Big(\frac{\psi(x)}{x}\Big)^{p-1}\psi'(x)\Big)'.$$

Note that it is nonnegative, since both $\psi(x)/x$ and $\psi'(x)$ are decreasing and $p-1>0$. For $0<a<x<b<\infty$ we have

$$\begin{aligned}
&\int_a^x w_0 + x^p\int_x^b t^{-p}w_0(t)dt \\
&= -\int_a^x t^p\Big(\Big(\frac{\psi(t)}{t}\Big)^{p-1}\psi'(t)\Big)'dt - x^p\int_x^b \Big(\Big(\frac{\psi(t)}{t}\Big)^{p-1}\psi'(t)\Big)'dt \\
&= -\Big(t^p\Big(\frac{\psi(t)}{t}\Big)^{p-1}\psi'(t)\Big)\Big|_a^x + p\int_a^x t^{p-1}\Big(\frac{\psi(t)}{t}\Big)^{p-1}\psi'(t)dt \\
&\quad - x^p\Big(\Big(\frac{\psi(t)}{t}\Big)^{p-1}\psi'(t)\Big|_x^b\Big) \\
&= a^p\Big(\frac{\psi(a)}{a}\Big)^{p-1}\psi'(a) - x^p\Big(\frac{\psi(b)}{b}\Big)^{p-1}\psi'(b) + \psi^p(x) - \psi^p(a).
\end{aligned}$$

The fact that $\psi'(t) = \widehat{\psi}(t)/t$ and other properties of $\widehat{\psi}$ and $\psi$ mentioned above enable us to take the limit as $a\to 0^+$ and $b\to\infty$ in the preceding calculation and to obtain that

$$\psi(x) = \Big(\int_0^x w_0 + x^p\int_x^\infty t^{-p}w_0(t)dt\Big)^{1/p}$$

for all $x>0$. In other words, $\varphi_{\Gamma_{p,w_0}}(x) = \psi(x)$ and so, by the remark immediately preceding Proposition 1.3, we have $\Gamma_{p,w_0} = \Gamma_{p,w}$. Since $w_0$ is continuous, the preceding formula for $\psi$ implies that

$$\begin{aligned}
x\psi'(x) &= x^p\Big(\int_0^x w_0 + x^p\int_x^\infty t^{-p}w_0(t)dt\Big)^{1/p-1}\int_x^\infty t^{-p}w_0(t)dt \\
&= x^p\psi^{1-p}(x)\int_x^\infty t^{-p}w_0(t)dt
\end{aligned}$$

for all $x>0$. But $x\psi'(x) = \widehat{\psi}(x) \approx \psi(x)$ and therefore, for some $A>0$,

$$A \le \frac{x^p\int_x^\infty t^{-p}w_0(t)dt}{\int_0^x w_0 + x^p\int_x^\infty t^{-p}w_0(t)dt} \le 1,$$

which yields that $\int_0^x w_0 \le Cx^p\int_x^\infty t^{-p}w_0(t)dt$, that is $w_0$ satisfies $RB_p$.

If $0<p\le 1$ then, as shown in the proof of Proposition 1.2(2), $\varphi'$ is decreasing. Moreover, for a.e. $x>0$,

$$x\varphi'(x) = x^p\int_x^\infty t^{-p}w(t)dt\cdot\varphi(x)^{1-p}.$$

In fact this extends to all $x > 0$, since our former calculation of $\varphi$ above, in terms of $W$, shows that $\varphi'$ exists and is continuous for all $x > 0$. This formula, together with the well known fact that for a decreasing function $u$, $U(x) = \int_0^x u$ is regular if and only if $xu(x) \approx U(x)$ (cf. [23]), quickly yields that the relation $x\varphi'(x) \approx \varphi(x)$ is equivalent to the inequality $\int_0^x w \leq Cx^p \int_x^\infty t^{-p}w(t)dt$ for some $C > 0$ and all $x > 0$. ∎

PROPOSITION 1.11: *If a weight function $w$ satisfies condition $RB_p$ and $\int_1^\infty w = \infty$, then for every $a > 0$,*

$$(C+1)^{-1/p}\overline{W}_p(1/a)^{1/p} \leq \|D_a\|_{\Gamma_{p,w}\to\Gamma_{p,w}} \leq \overline{W}_p(1/a)^{1/p},$$

*where $C$ is the constant appearing in condition $RB_p$, and*

$$\overline{W_p}(1/a) = \sup_{t>0} W_p(t/a)/W_p(t).$$

*Consequently, the Boyd indices of $\Gamma_{p,w}$ are as follows:*

$$p(\Gamma_{p,w}) = p/\beta(W_p) \quad \text{and} \quad q(\Gamma_{p,w}) = p/\alpha(W_p).$$

*Proof:* For any $a > 0$ we get

$$\begin{aligned}\|D_a\| :=& \|D_a\|_{\Gamma_{p,w}\to\Gamma_{p,w}} \geq \sup_{t>0}(\varphi(t/a)/\varphi(t))\\ &\geq \sup_{t>0}(W_p(t/a)/(C+1)W_p(t))^{1/p} = (\overline{W}_p(1/a)/(C+1))^{1/p}.\end{aligned}$$

On the other hand, by the second formula for the quasi-norm in Proposition 1.1,

$$\begin{aligned}\|D_a f\|^p &= \int_0^\infty t^{-p}W_p(t)d\bigg(\Big(\int_0^t f^*(as)ds\Big)^p\bigg)\\ &= a^{-p}\int_0^\infty (z/a)^{-p}W_p(z/a)d\bigg(\Big(\int_0^z f^*(u)du\Big)^p\bigg)\\ &\leq \overline{W}_p(1/a)\int_0^\infty z^{-p}W_p(z)d\bigg(\Big(\int_0^z f^*(u)du\Big)^p\bigg) = \overline{W}_p(1/a)\|f\|^p,\end{aligned}$$

which in combination with the previous inequality gives the required estimation. ∎

COROLLARY 1.12: I. *If $0 < p \leq 1$ then $w$ satisfies condition $RB_p$ if and only if $q(\Gamma_{p,w}) < \infty$.*

II. *Let $1 < p < \infty$. If $w$ satisfies condition $RB_p$ then $q(\Gamma_{p,w}) < \infty$. Conversely, if $q(\Gamma_{p,w}) < \infty$ then there exists a weight function $w_0$ such that $\Gamma_{p,w} = \Gamma_{p,w_0}$ and $w_0$ satisfies $RB_p$.*

PROPOSITION 1.13: *Let $E$ be a r.i. quasi-Banach function space. If the fundamental function of $E$ is not regular, i.e. $\alpha(\varphi_E) = 0$, then $E$ contains order copies of $\ell_n^\infty$ uniformly for $n \in \mathbb{N}$.*

*Proof:* We shall show first that if $F\colon (0,\infty) \to (0,\infty)$ is an increasing function and for some $b > 1$,

$$D := \inf_{x>0} \frac{F(bx)}{F(x)} > 1,$$

then $\alpha(F) > 0$. In fact, the above inequality implies that $F(b^{-1}x) \le D^{-1}F(x)$, $x > 0$. Let $0 < a \le 1$ be fixed. Then there exists $n = 0, 1, \ldots$ such that $1/b^{n+1} < a \le 1/b^n$. Setting $q = \ln D / \ln b$ we have for any $x > 0$

$$F(ax) \le F(b^{-n}x) \le D^{-n}F(x) = (b^q)^{-n}F(x) = (1/b^{n+1})^q b^q F(x) \le b^q a^q F(x),$$

which proves that $\alpha(F) > 0$.

Thus, if $\varphi_E$ is not regular then for every $n \in \mathbb{N}$

$$\inf_{x>0} \frac{\varphi_E(nx)}{\varphi_E(x)} = 1.$$

Hence for fixed $n \in \mathbb{N}$ and $\epsilon > 0$ there exists $\alpha > 0$ such that

$$\varphi_E(n\alpha) \le (1+\epsilon)\varphi_E(\alpha).$$

Defining $f_k = (\varphi_E(\alpha))^{-1}\chi_{((k-1)\alpha, k\alpha)}$ for $k = 1, \ldots, n$, we obtain

$$1 = \|f_k\| \le \Big\|\sum_{k=1}^n f_k\Big\| = \frac{\varphi_E(n\alpha)}{\varphi_E(\alpha)} \le 1 + \epsilon,$$

which shows that $\{f_1, \ldots, f_n\}$ (depending on $n$) span copies of $\ell_n^\infty$ in $E$ uniformly for $n \in \mathbb{N}$. ∎

## 2. Copies of $\ell^p$

For $1 \le p < \infty$ and under some additional assumptions on the weight $w$, the fact that the space $\Gamma_{p,w}$ contains an isomorphic copy of $\ell^p$ follows from the well known theorem of Levy [27] (see also [5]), that the Lions–Peetre interpolation space $(A_0, A_1)_{\theta,p}$ contains an isomorphic copy of $\ell^p$. The next theorem extends this result to any $0 < p < \infty$ and to arbitrary weight.

THEOREM 2.1: *Let $0 < p < \infty$ and $w$ be a weight function such that $\Gamma_{p,w}$ is a proper subspace of $L^1 + L^\infty$, that is either $\int_1^\infty w = \infty$ or $\int_0^1 t^{-p} w(t)dt = \infty$. Then $\Gamma_{p,w}$ contains an order almost isometric and complemented copy of $\ell^p$.*

*Proof:* We shall define a sequence $(f_n)$ of disjointly supported functions in $\Gamma_{p,w}$ such that they span an isomorphic copy of $\ell^p$ in $\Gamma_{p,w}$. Since the $f_n$ will be constructed so that they are constant on their supports:$=A_n$, the copy of $\ell^p$ will be complemented in $\Gamma_{p,w}$. In fact, the averaging operator $P$ defined by

$$Pf = \sum_{n=1}^{\infty} \Big( \frac{1}{|A_n|} \int_{A_n} f \Big) \chi_{A_n}$$

is a bounded projection of $\Gamma_{p,w}$ onto the closed span of $\{f_n\}$, since $\Gamma_{p,w}$ is an interpolation space between $L^1$ and $L^\infty$. Let $\epsilon > 0$ be an arbitrary number.

We start with $0 < p < 1$ and the case of $\int_1^\infty w = \infty$. First we choose numbers $b_1, d_1 > 0$ such that $\|f_1\| = 1$ where

$$f_1 = b_1 \chi_{(0,d_1)}.$$

Then let $0 < c_1 < d_1$ and $c_2 > d_1$ be such that

$$\int_0^{c_1} f_1^{**p} w < \epsilon/2 \quad \text{and} \quad \int_{c_2}^{\infty} f_1^{**p} w < \epsilon/2.$$

Next we choose $0 < b_2 < b_1$ and $d_2 > c_2 + d_1$ such that setting

$$f_2 = b_2 \chi_{(d_1,d_2)},$$

we get

$$\|f_2\| = 1 \quad \text{and} \quad \int_0^{c_2} f_2^{**p} w < \epsilon/2^2.$$

We then find $c_3 > d_2$ with

$$\int_{c_3}^{\infty} (f_1 + f_2)^{**p} < \epsilon/2^2.$$

It is clear now that by induction we shall find sequences $(b_j), (c_j)$ and $(d_j)$ of positive numbers and a sequence of functions $(f_j)$ such that for $j = 1, 2, \ldots$, $0 = c_0 = d_0 < c_1 < d_1 < \cdots < c_j < d_j$, $d_j - d_{j-1} > c_j$ and

$$f_j = b_j \chi_{(d_{j-1},d_j)}$$

with $\|f_j\| = 1$ and

$$\int_0^{c_j} f_j^{**p} w < \epsilon/2^j \quad \text{and} \quad \int_{c_{j+1}}^{\infty} \left(\sum_{i=1}^{j} f_i\right)^{**p} w < \epsilon/2^j.$$

Note here that both functions $\sum_{i=1}^{\infty} |a_i f_i| := \sup_i\{|a_i f_i|\}$ and $(\sum_{i=1}^{\infty} a_i f_i)^{**}$ are well defined. Letting now $(a_i)$ in $\ell^p$ with $\|(a_i)\|_{\ell^p} = 1$ and $a_0 = 0$ we obtain

$$\begin{aligned}\left\|\sum_{i=1}^{\infty} a_i f_i\right\|^p &= \sum_{j=0}^{\infty} \int_{c_j}^{c_{j+1}} \left(\sum_{i\neq j} a_i f_i + a_j f_j\right)^{**p} w \\ &\le \sum_{j=0}^{\infty} \int_{c_j}^{c_{j+1}} \left(\sum_{i\neq j} a_i f_i\right)^{**p} w + \sum_{j=0}^{\infty} \int_{c_j}^{c_{j+1}} (a_j f_j)^{**p} w.\end{aligned}$$

It is clear that

$$\sum_{j=0}^{\infty} \int_{c_j}^{c_{j+1}} (a_j f_j)^{**p} w \le \sum_{j=0}^{\infty} |a_j|^p \int_0^{\infty} f_j^{**p} w \le 1.$$

We also have for $j = 0, 1, \ldots,$

$$\int_{c_j}^{c_{j+1}} \left(\sum_{i\neq j} a_i f_i\right)^{**p} w \le \int_{c_j}^{c_{j+1}} \left(\sum_{i\neq j} f_i\right)^{**p} w.$$

Now, adopting the convention that $\sum_{\emptyset} = 0$, we obtain for any $j = 0, 1, \ldots$

$$\int_{c_j}^{c_{j+1}} \left(\sum_{i\neq j} f_i\right)^{**p} w = \int_{c_j}^{c_{j+1}} \left(\sum_{i=1}^{j-1} f_i + f_{j+1}\right)^{**p} w.$$

Since the construction of $(f_n)$ yields

$$\left(\sum_{i=1}^{j-1} f_i + f_{j+1}\right)^{*} = \sum_{i=1}^{j-1} b_i \chi_{(d_{i-1}, d_i)} + b_{j+1} \chi_{(d_{j-1}, d_{j-1}+d_{j+1}-d_j)}$$

and $c_{j+1} < d_{j+1} - d_j < d_{j+1} - d_j + d_{j-1}$, so for every $j = 0, 1, \ldots$

$$\begin{aligned}\int_{c_j}^{c_{j+1}} \left(\sum_{i\neq j} f_i\right)^{**p} w &\le \int_{c_j}^{\infty} \left(\sum_{i=1}^{j-1} f_i\right)^{**p} w + \int_0^{c_{j+1}} f_{j+1}^{**p} w \\ &\le \epsilon/2^{j-1} + \epsilon/2^{j+1} \le \epsilon/2^{j-2}.\end{aligned}$$

Combining finally the above estimations we get

$$\left\|\sum_{i=1}^{\infty} a_i f_i\right\| \le (1 + 8\epsilon)^{1/p}.$$

On the other hand, for any sequence $(a_i)$ in $\mathbb{R}$,

$$\Big\|\sum_{i=1}^{\infty} a_i f_i\Big\| \ge \Big(\sum_{j=1}^{\infty} |a_j|^p \int_{c_j}^{c_{j+1}} f_j^{**p} w\Big)^{1/p},$$

and in view of the inequality,

$$\int_{c_j}^{c_{j+1}} f_j^{**p} w = 1 - \int_0^{c_j} f_j^{**p} w - \int_{c_{j+1}}^{\infty} f_j^{**p} w \ge 1 - \epsilon/2^{j-1},$$

we have

$$\Big\|\sum_{i=1}^{\infty} a_i f_i\Big\| \ge \Big(\sum_{j=1}^{\infty} |a_j|^p (1-\epsilon/2^{j-1})\Big)^{1/p} \ge (1-\epsilon)^{1/p} \|(a_i)\|_{\ell^p}.$$

Assume now that $0 < p < 1$ and

$$\infty = \Big(\int_0^1 t^{-p} w(t)dt\Big)^{1/p} = \Big(\int_0^{\infty} t^{-p} w(t)dt\Big)^{1/p} = \lim_{x\to 0+} \varphi(x)/x.$$

Then the sequence $(f_n)$ will be defined as follows. Let $d_0 = k_1 = 1, d_1 = 1/2^{k_1}$ and choose $b_1 > 0$ such that $\|f_1\| = 1$, where $f_1 = b_1\chi_{(d_1,d_0)}$. There exists $\mathbb{N} \ni n_1 > k_1$ such that, setting $c_1 = 1/2^{n_1}$, we obtain

$$\int_0^{c_1} f_1^{**p} w < \epsilon/2.$$

Now for any $\mathbb{N} \ni k > n_1$ set $d_1 - x_k = 1/2^k$. Since $(d_1 - x_k)/\varphi(d_1 - x_k) \to 0$ as $k \to \infty$, there exists $k_2 > n_1$ such that

$$\Big(\frac{d_1 - x_{k_2}}{\varphi(d_1 - x_{k_2})}\Big)^p \int_{1/2^{n_1}}^{\infty} t^{-p} w(t)dt < \epsilon/2^2.$$

Letting then $d_2 = d_1 - 1/2^{k_2}$ and $b_2 = 1/\varphi(d_1 - d_2)$ and $f_2 = b_2\chi_{(d_2,d_1)}$ we obtain clearly that $\|f_2\| = 1$. Moreover,

$$\begin{aligned}\int_{c_1}^{\infty} f_2^{**p} w &= \int_{1/2^{n_1}}^{\infty} (b_2\chi_{(0,d_1-d_2)})^{**p} w \\ &= \Big(\frac{d_1 - d_2}{\varphi(d_1 - d_2)}\Big)^p \int_{1/2^{n_1}}^{\infty} t^{-p} w(t)dt < \epsilon/2^2.\end{aligned}$$

Next we find $\mathbb{N} \ni n_2 > k_2$ such that, setting $c_2 = 1/2^{n_2}$,

$$\int_0^{c_2} (f_1 + f_2)^{**p} w < \epsilon/2^2.$$

Consequently, by induction process we find sequences of natural numbers $(n_j),(k_j)$ and of positive numbers $(b_j),(d_j)$ such that for $j = 1,2,\ldots$, $1 = d_0 = k_1 < n_1 < k_2 < \cdots < k_j < n_j < k_{j+1}$, $d_{j-1} - d_j = 1/2^{k_j}$ and $\|f_j\| = 1$ where

$$f_j = b_j \chi_{(d_j,d_{j-1})}.$$

Moreover, for $c_j = 1/2^{n_j}$,

$$\int_{c_j}^{\infty} f_{j+1}^{**p} w < \epsilon/2^{j+1} \quad \text{and} \quad \int_0^{c_j} \Big(\sum_{i=1}^{j} f_i\Big)^{**p} w < \epsilon/2^j.$$

Now, setting $c_0 = \infty$ and once again using the convention that $\sum_{\emptyset} = 0$, we obtain for any $(a_i) \in \ell^p$ with $\|(a_i)\|_{\ell^p} = 1$, and any $m \in \mathbb{N}$,

$$\Big\|\sum_{i=1}^{m} a_i f_i\Big\| \le \Big(1 + \sum_{j=0}^{\infty} \int_{c_{j+1}}^{c_j} \Big(\sum_{\substack{i \ne j+1 \\ i=1}}^{m} a_i f_i\Big)^{**p} w\Big)^{1/p}.$$

Moreover, for any $j = 0, 1, \ldots$,

$$\begin{aligned} \int_{c_{j+1}}^{c_j} \Big(\sum_{\substack{i \ne j+1 \\ i=1}}^{m} a_i f_i\Big)^{**p} w &\le \sum_{i=j+2}^{m} \Big(\int_{c_{j+1}}^{c_j} f_i^{**p} w\Big) + \int_{c_{j+1}}^{c_j} \Big(\sum_{i=1}^{j} f_i\Big)^{**p} w \\ &\le \sum_{i=j+2}^{m} \Big(\int_{c_{i-1}}^{\infty} f_i^{**p} w\Big) + \int_0^{c_j} \Big(\sum_{i=1}^{j} f_i\Big)^{**p} w \\ &\le \sum_{i=j+2}^{\infty} \frac{\epsilon}{2^i} + \frac{\epsilon}{2^j} \le \frac{\epsilon}{2^{j-2}}. \end{aligned}$$

Thus, for every $m \in \mathbb{N}$,

$$\Big\|\sum_{i=1}^{m} a_i f_i\Big\| \le (1 + 8\epsilon)^{1/p}.$$

On the other hand, for any sequence $(a_i) \subset \mathbb{R}$ and $m \in \mathbb{N}$,

$$\Big\|\sum_{i=1}^{m} a_i f_i\Big\| \ge (1 - \epsilon)^{1/p} \Big(\sum_{i=1}^{m} |a_i|^p\Big)^{1/p}.$$

In case of $1 \le p < \infty$ we shall only sketch the proof when $\int_0^\infty w = \infty$. By induction we find a disjointly supported normalized sequence of functions $(f_j = b_j \chi_{(d_{j-1},d_j)}) \subset \Gamma_{p,w}$ with $(b_j)$ positive, decreasing and satisfying

$$\Big(\int_{c_{j+1}}^{\infty} f_j^{**p} w\Big)^{1/p} < \epsilon/4^j \quad \text{and} \quad \Big(\int_0^{c_j} f_j^{**p} w\Big)^{1/p} < \epsilon/4^j, \quad j = 0, 1 \ldots,$$

where $0 = c_0 < c_1 < \cdots < c_j < c_{j+1} < \cdots$. Let $(a_i)$ belong to the unit sphere of $\ell^p$. By applying Minkowski's inequality twice we obtain

$$\left\|\sum_{i=1}^{\infty} a_i f_i\right\| \leq \left(\sum_{j=0}^{\infty}\int_{c_j}^{c_{j+1}} (a_j f_j)^{**p} w\right)^{1/p} + \left(\sum_{j=0}^{\infty}\int_{c_j}^{c_{j+1}} \left(\sum_{i\neq j} a_i f_i\right)^{**p} w\right)^{1/p}.$$

Moreover, again by Minkowski's inequality,

$$\begin{aligned}\left(\sum_{j=0}^{\infty}\int_{c_j}^{c_{j+1}} \left(\sum_{i\neq j} a_i f_i\right)^{**p} w\right)^{1/p} &\leq \sum_{i=1}^{\infty}\left(\sum_{j\neq i}\int_{c_j}^{c_{j+1}} f_i^{**p} w\right)^{1/p} \\ &= \sum_{i=1}^{\infty}\left(\int_0^{c_i} f_i^{**p} w + \int_{c_{i+1}}^{\infty} f_i^{**p} w\right)^{1/p} \\ &\leq \sum_{i=1}^{\infty}\left(\int_0^{c_i} f_i^{**p} w\right)^{1/p} + \sum_{i=1}^{\infty}\left(\int_{c_{i+1}}^{\infty} f_i^{**p} w\right)^{1/p} \\ &\leq 2\sum_{i=1}^{\infty}\frac{\epsilon}{4^i} < \epsilon.\end{aligned}$$

Hence

$$\left\|\sum_{i=1}^{\infty} a_i f_i\right\| \leq 1 + \epsilon.$$

Finally, for any $(a_j) \subset \mathbb{R}$,

$$\left\|\sum_{i=1}^{\infty} a_i f_i\right\| \geq \left(\sum_{j=1}^{\infty}\int_{c_j}^{c_{j+1}} (a_j f_j)^{**p} w\right)^{1/p} = (1-\epsilon)\|(a_i)\|_{\ell^p},$$

and the proof is complete. ∎

## 3. Convexity and concavity of $\Gamma_{p,w}$

In this section we provide criteria for order convexity and concavity as well as for lower and upper estimates in $\Gamma_{p,w}$. For $1 \leq p < \infty$ we also state criteria on type and cotype of $\Gamma_{p,w}$. These results are expressed equivalently in terms of integral inequalities, indices and growth conditions of either the fundamental function $\varphi$ or $W_p$. We start with a simple but useful lemma.

LEMMA 3.1: *If a r.i. quasi-Banach function space $(E, \|\cdot\|_E)$ satisfies an upper (resp. lower) $r$-estimate, $0 < r < \infty$, then $\varphi_E(x)/x^{1/r}$ is pseudo-decreasing (resp. pseudo-increasing).*

*Proof:* Assume that $E$ satisfies an upper $r$-estimate. Then for any $0 < s < t$, $n = [t/s]$ and

$$f_i = \chi_{\left(\frac{(i-1)t}{2n}, \frac{it}{2n}\right]}, \quad i = 1, \ldots, 2n,$$

we obtain

$$\begin{aligned}\varphi_E(t) &= \Big\| \sum_{i=1}^{2n} |f_i| \Big\|_E \le C\Big(\sum_{i=1}^{2n} \|f_i\|_E^r\Big)^{1/r} = C\Big(\sum_{i=1}^{2n} \varphi_E^r(t/2n)\Big)^{1/r} \\ &= C2^{1/r} n^{1/r} \varphi_E(t/2n) \le C2^{1/r}(t/s)^{1/r}\varphi_E(s),\end{aligned}$$

which simply yields that $\varphi_E(t)/t^{1/r} \le K\varphi_E(s)/s^{1/r}$, where $K = C2^{1/r}$. ∎

THEOREM 3.2: *Let $1 < p < \infty$ and $w$ be a weight function satisfying condition $RB_p$ and $\int_0^1 t^{-p}w(t)dt = \int_1^\infty w = \infty$. Then the following conditions are equivalent.*

(i) *$\Gamma_{p,w}$ is $p$-convex (resp. $p$-concave).*
(ii) *$\Gamma_{p,w}$ satisfies an upper $p$-estimate (resp. a lower $p$-estimate).*
(iii) *$W_p(x)/x$ is pseudo-decreasing (resp. pseudo-increasing).*
(iv) *$\varphi(x)/x^{1/p}$ is pseudo-decreasing (resp. pseudo-increasing).*

*Proof:* In view of Theorem 1.8, there exists a weight $v$ such that $\Lambda_{p',v}$ is a normable space and it is a predual of $\Gamma_{p,w}$, that is $\Lambda^*_{p',v}$ is lattice isomorphic to $\Gamma_{p,w}$. Thus $\Gamma_{p,w}$ is $p$-convex or satisfies an upper $p$-estimate if and only if $\Lambda_{p',v}$ is $p'$-concave or satisfies a lower $p'$-estimate, respectively (cf. Proposition 1.d.4 in [28]). Moreover, by Theorem 1.8, $V(x) = (x^{-p}W_p(x))^{-1/(p-1)}$. Hence $V(x)/x$ is pseudo-increasing if and only if $W_p(x)/x$ is pseudo-decreasing. Now, applying Theorem 8 in [22] to $\Lambda_{p',v}$, conditions (i)–(iii) are equivalent. Finally, condition $RB_p$ yields that $\varphi^p \approx W_p$, and hence (iii) is equivalent to (iv). The parallel conditions in parentheses are also handled by duality and the appropriate results in Lorentz space $\Lambda_{p',v}$ ([22], Theorem 4). ∎

THEOREM 3.3: *Let $1 < p < \infty$ and $w$ be a weight function satisfying condition $RB_p$ and $\int_0^1 t^{-p}w(t)dt = \int_1^\infty w = \infty$.*

*If $r > p$ (resp. $1 \le r < p$) then $\Gamma_{p,w}$ does not satisfy an upper (resp. lower) $r$-estimate.*

*For $1 < r \le p$ (resp. $r \ge p$) the following conditions are equivalent.*

(i) $\Gamma_{p,w}$ *satisfies an upper $r$-estimate (resp. a lower $r$-estimate).*

(ii) $W_p(x)/x^{p/r}$ *is pseudo-decreasing (resp. pseudo-increasing).*

(iii) $\varphi(x)/x^{1/r}$ *is pseudo-decreasing (resp. pseudo-increasing).*

*Proof:* By Theorem 2.1, $\Gamma_{p,w}$ contains a copy of $\ell^p$. So if $r > p$ (resp. $1 < r < p$) then $\Gamma_{p,w}$ does not satisfy an upper (resp. a lower) $r$-estimate. The remaining part of the proof is obtained by the duality method analogously to the proof of the previous theorem with applications of Theorems 3 and 7 of [22]. ∎

THEOREM 3.4: *Let $1 < p < \infty$ and $w$ be a weight function satisfying condition $RB_p$ and $\int_0^1 t^{-p}w(t)dt = \int_1^\infty w = \infty$.*

*If $r > p$ (resp. $1 \le r < p$) then $\Gamma_{p,w}$ is not $r$-convex (resp. $r$-concave)*

*For $1 < r < p$ (resp. $r > p$) the following conditions are equivalent.*

(i) $\Gamma_{p,w}$ *is $r$-convex (resp. $r$-concave).*

(ii) $\beta(W_p) < p/r$ *(resp. $\alpha(W_p) > p/r$), or equivalently for some $\epsilon > 0$, $W_p(x)/x^{p/r-\epsilon}$ (resp. $W_p(x)/x^{p/r+\epsilon}$) is pseudo-decreasing (resp. pseudo-increasing).*

(iii) $\beta(\varphi) < 1/r$ *(resp. $\alpha(\varphi) > 1/r$).*

(iv) *The Hardy operator $H^r$ (resp. $H_r$) is bounded on $\Gamma_{p,w}$.*

(v) *There exists $C > 0$ such that for all $x > 0$,*

$$\int_x^\infty t^{-p/r}w(t)dt \le Cx^{-p/r}W_p(x)$$

$$\left(\text{resp. } \int_0^x t^{-p/r}w(t)dt \le Cx^{-p/r}W_p(x)\right).$$

*Proof:* Applying Theorem 1.8 and appropriate results on convexity and concavity in $\Lambda_{p,w}$ ([22], Theorems 2 and 6) we can prove the equivalence of conditions (i)–(iii) by duality. For instance, we shall show the equivalence of (i) and (ii). If $\Gamma_{p,w}$ is $r$-convex then $\Lambda_{p',v}$ is $r'$-concave, which in turn is equivalent to $\alpha(V) > p'/r'$ by Theorem 6 in [22], where $V(x) = (x^{-p}W_p(x))^{-1/(p-1)}$ is a function which appears in the statement of Theorem 1.8. Hence, by properties of indices,

$$\alpha(V) = -\frac{1}{p-1}\beta(x^{-p}W_p(x)) = \frac{p}{p-1} - \frac{1}{p-1}\beta(W_p) > \frac{p}{p-1}\frac{r-1}{r},$$

and so $\beta(W_p) < p/r$.

The equivalence of (ii) and (iv) follows from Proposition 1.11 and the well known fact that $p(E) > r$ (resp. $q(E) < r$) if and only if $H^r$ (resp. $H_r$) is bounded on $E$ [31].

Finally, we shall show equivalence of conditions (ii) and (v). Let us start with conditions corresponding to convexity. By Fubini's theorem,

$$\int_x^\infty t^{-p/r} W_p(t)\frac{dt}{t} = \int_x^\infty \left(\int_x^s t^{-p/r+p-1}dt\right) s^{-p}w(s)ds$$
$$= \frac{r'}{p}\left(\int_x^\infty s^{-p/r}w(s)ds - x^{-p/r}W_p(x)\right).$$

If we assume that $\beta(W_p) < p/r$, then $W_p(x)/x^{p/r-\epsilon}$ is pseudo-decreasing for some $\epsilon > 0$. Hence for any $x > 0$,

$$\int_x^\infty t^{-p/r}W_p(t)\frac{dt}{t} \le C\frac{W_p(x)}{x^{p/r-\epsilon}}\int_x^\infty t^{-\epsilon-1}dt = \frac{C}{\epsilon}x^{-p/r}W_p(x).$$

Thus

$$\int_x^\infty t^{-p/r}w(t)dt \le ((Cp/\epsilon r') + 1)x^{-p/r}W_p(x).$$

Assuming now the first inequality in (v), we obtain

$$\int_x^\infty t^{-p/r}w(t)dt = \frac{p}{r'}\int_x^\infty t^{-p/r}W_p(t)\frac{dt}{t} + x^{-p/r}W_p(x) \le Cx^{-p/r}W_p(x).$$

Hence

$$\int_x^\infty t^{-p/r}W_p(t)\frac{dt}{t} \le (C-1)\frac{r'}{p}x^{-p/r}W_p(x).$$

Applying then Theorem 6.4 in [30], $\beta(W_p) < p/r$.

Now, let $\alpha(W_p) > p/r$. Then $W_p(x)/x^{p/r+\epsilon}$ is pseudo-increasing for some $\epsilon > 0$. Since $W_p(x)/x^p$ is also decreasing, it follows, for all $x > 0$, that

$$\frac{r'}{p}x^{-p/r}W_p(x) \le \int_0^x t^{-p/r}W_p(t)\frac{dt}{t} \le \frac{C}{\epsilon}x^{-p/r}W_p(x),$$

which together with the equality

$$\int_0^x t^{-p/r}W_p(t)\frac{dt}{t} = \frac{r'}{p}\left(\int_0^x s^{-p/r}w(s)ds + x^{-p/r}W_p(x)\right)$$

yields

$$\int_0^x t^{-p/r}W_p(t)\frac{dt}{t} \le ((Cp/\epsilon r') - 1)x^{-p/r}W_p(x).$$

On the other hand, assuming the inequality in parentheses in (iv), we obtain, for all $x > 0$,

$$\int_0^x t^{-p/r}w(t)dt = \frac{p}{r'}\int_0^x t^{-p/r}W_p(t)\frac{dt}{t} - x^{-p/r}W_p(x) \le Cx^{-p/r}W_p(x),$$

which yields

$$\int_0^x t^{-p/r} W_p(t) \frac{dt}{t} \le (C+1) \frac{r'}{p} x^{-p/r} W_p(x).$$

Applying again Theorem 6.4 in [30], $\alpha(W_p) > p/r$, and the proof is finished. ∎

In view of Proposition 1.4, the assumptions $\int_0^1 t^{-p} w(t)dt = \int_1^\infty w = \infty$ imposed in Theorems 3.2–3.4 exclude only a somewhat marginal situation when either $L^\infty$ or $L^1$ is contained in $\Gamma_{p,w}$.

THEOREM 3.5: *Let $0 < p \le 1$.*

I. *$\Gamma_{p,w}$ is $p$-convex, so it satisfies an upper $p$-estimate.*

II. *If $\Gamma_{p,w}$ does not coincide with $L^1 + L^\infty$, that is $\int_0^1 t^{-p} w(t)dt = \infty$ or $\int_1^\infty w = \infty$, then $\Gamma_{p,w}$ does not satisfy an upper $r$-estimate and so it is not $r$-convex for any $r > p$.*

*Proof:* Observe that for $q = 1/p$, the set $E$ of all $f \in L^0$ with

$$\|f\|_E := \int_0^\infty \left( \frac{1}{t} \int_0^t f^{*q}(s)ds \right)^{1/q} w(t)dt < \infty$$

is a Banach function space, such that its $p$-convexification coincides with $\Gamma_{p,w}$. Hence $\Gamma_{p,w}$ is $p$-convex. Under the assumptions in II, $\Gamma_{p,w}$ does not satisfy an upper $r$-estimate for $r > p$, since it contains an order copy of $\ell^p$ by Theorem 2.1. ∎

The next two theorems on lower estimates and concavity of $\Gamma_{p,w}$ in case when $0 < p \le 1$ have the same formulation as their corresponding parts for $1 < p < \infty$ (cf. Theorems 3.3, 3.4), but their proofs are different, since they cannot be handled by duality.

First we make the following observation. If $\int_0^1 t^{-p} w(t)dt = \infty$, then $\Gamma_{p,w}$ does not satisfy a lower 1-estimate and so it cannot be 1-concave. Indeed, since $\varphi(x)/x$ is decreasing and $\lim_{x\to 0^+} \varphi(x)/x = (\int_0^\infty t^{-p} w(t)dt)^{1/p} = \infty$ by Proposition 1.4(4), the fundamental function $\varphi$ does not satisfy the condition in Lemma 3.1, and the conclusion follows.

THEOREM 3.6: *Let $0 < p \le 1$ and $\int_0^1 t^{-p} w(t)dt = \infty$.*

I. *Given $0 < r < \infty$, if either $w$ does not satisfy condition $RB_p$ or $0 < r < p$ or $0 < r \le 1$, then $\Gamma_{p,w}$ does not satisfy a lower $r$-estimate.*

II. *Consider the following conditions.*

(i) *$\Gamma_{p,w}$ satisfies a lower $r$-estimate.*

(ii) $W_p(x)/x^{p/r}$ *is pseudo-increasing.*

(iii) $\varphi(x)/x^{1/r}$ *is pseudo-increasing.*

*If $w$ satisfies condition $RB_p$ and $r \geq p$, then (ii) and (iii) are equivalent and (i) implies (ii).*

*Proof:* Part I is a consequence of Propositions 1.10, 1.13, Theorem 2.1 and the observation made before the theorem. The conditions (ii) and (iii) of part II are equivalent in view of the assumption that condition $RB_p$ holds. The implication from (i) to (ii) easily follows from Lemma 3.1. ∎

THEOREM 3.7: *Let $0 < p \leq 1$ and $\int_0^1 t^{-p} w(t)dt = \infty$.*

I. *Given $0 < r < \infty$, if either $w$ does not satisfy condition $RB_p$ or $0 < r < p$ or $0 < r \leq 1$, then $\Gamma_{p,w}$ is not $r$-concave.*

II. *If $w$ satisfies condition $RB_p$ and $r \geq \max(p,1)$, then the following conditions are equivalent.*

(i) $\Gamma_{p,w}$ *is $r$-concave.*

(ii) $\alpha(W_p) > p/r$, *or equivalently for some $\epsilon > 0$, $W_p(x)/x^{p/r+\epsilon}$ is pseudo-increasing.*

(iii) $\alpha(\varphi) > 1/r$.

(iv) *The Hardy operator $H_r$ is bounded on $\Gamma_{p,w}$.*

(v) *There exists $C > 0$ such that for all $x > 0$,*

$$\int_0^x t^{-p/r} w(t)dt \leq C x^{-p/r} W_p(x).$$

*Proof:* Part I is a result of Theorem 3.6(I). In order to prove part II, we first observe that the equivalence of (ii), (iii) and (v) can be shown in the same way as the corresponding conditions in Theorem 3.4. Assuming now condition (iv), that $H_r$ is bounded on $\Gamma_{p,w}$, we shall show that $\Gamma_{p,w}$ is $r$-concave, which is (i). Recall that

$$H_r\left(\left(\sum_{i=1}^n |f_i|^r\right)^{1/r}\right) \geq \left(\sum_{i=1}^n H_r(|f_i|)^r\right)^{1/r},$$

and that $H_r f(x) \geq f^*(2x)$. Thus for any $f_i \in \Gamma_{p,w}$, $i = 1, \ldots, n$,

$$C\left\|\left(\sum_{i=1}^n |f_i|^r\right)^{1/r}\right\| \geq \left\|H_r\left(\left(\sum_{i=1}^n |f_i|^r\right)^{1/r}\right)\right\| \geq \left\|\left(\sum_{i=1}^n H_r(|f_i|)^r\right)^{1/r}\right\|$$

$$\geq \left(\int_0^\infty \left(\frac{1}{t}\int_0^t \left(\sum_{i=1}^n f_i^{*r}(2s)\right)^{1/r} ds\right)^p w(t)dt\right)^{1/p}$$

$$= \left\|\left(\sum_{i=1}^n f_i^{*r}(2\cdot)\right)^{1/r}\right\| := A.$$

But for $r > p$ the weighted $L^p(w)$ space is $r$-concave, so for $g_i(t) = H^1(f_i^*)(t)$, in view of Minkowski's inequality applied for $r > 1$, we have

$$\begin{aligned}\Big(\sum_{i=1}^n \|f_i\|^r\Big)^{1/r} &= \Big(\sum_{i=1}^n \|g_i\|^r_{L^p(w)}\Big)^{1/r} \le C\Big\|\Big(\sum_{i=1}^n |g_i|^r\Big)^{1/r}\Big\|_{L^p(w)} \\ &= C\Big(\int_0^\infty \Big(\frac{1}{t}\Big\|\Big(\int_0^t f_i^*(s)ds\Big)_{i=1}^n\Big\|_{\ell^r}\Big)^p w(t)dt\Big)^{1/p} \\ &\le C\Big(\int_0^\infty \Big(\frac{1}{t}\int_0^t \|(f_i^*(s))_{i=1}^n\|_{\ell^r} ds\Big)^p w(t)dt\Big)^{1/p} \\ &= C\Big(\int_0^\infty \Big(\frac{1}{t}\int_0^t \Big(\sum_{i=1}^n f_i^{*r}(s)\Big)^{1/r} ds\Big)^p w(t)dt\Big)^{1/p} \\ &= C\Big\|\Big(\sum_{i=1}^n f_i^{*r}(\cdot)\Big)^{1/r}\Big\| := B.\end{aligned}$$

Now, since the dilation operator is bounded on any r.i. quasi-Banach space (cf. [16], Proposition 2; cf. also the proof of Theorem 4.4 in [25]), we have that $\|f(\cdot)\| \le C\|f(2\cdot)\|$ by Proposition 1.11. Thus $B \le CA$, and so $r$-concavity of $\Gamma_{p,w}$ has been proved.

In order to finish we need to show that (i) implies (v). We first observe that the $r$-concavity of $\Gamma_{p,w}$ yields the inequality

$$\Big\|\Big(\int_a^b |F(t,\cdot)|^r dt\Big)^{1/r}\Big\| \ge C\Big(\int_a^b \|F(t,\cdot)\|^r dt\Big)^{1/r},$$

for any $F(t,\cdot) \in \Gamma_{p,w}, 0 < a < b < \infty$ and some $C > 0$. For $x, y > 0$ and $f \in L^0$, define

$$F_r(y) = \Big(\int_0^x f^{*r}(|y-t|)dt\Big)^{1/r} \chi_{(0,x)}(y).$$

We have

$$\begin{aligned}F_r(y) &= \Big(\int_0^y f^{*r}(y-t)dt + \int_y^x f^{*r}(t-y)dt\Big)^{1/r} \chi_{(0,x)}(y) \\ &= \Big(\int_0^y f^{*r}(s)ds + \int_0^{x-y} f^{*r}(s)ds\Big)^{1/r} \chi_{(0,x)}(y) \\ &\le \Big(2\int_0^x f^{*r}(s)ds\Big)^{1/r} \chi_{(0,x)}(y).\end{aligned}$$

Hence

$$F_r^{**}(y) \le \Big(2\int_0^x f^{*r}(s)ds\Big)^{1/r} \chi_{(0,x)}^{**}(y),$$

and so

$$\|F_r\| \leq \left(2\int_0^x f^{*r}(s)ds\right)^{1/r} \varphi(x).$$

Thus, $r$-concavity of $\Gamma_{p,w}$ implies

$$\|F_r\| = \left\| \left( \int_0^x f^{*r}(|y-t|)dt\chi_{(0,x)}(y)\right)^{1/r} \right\| \geq C\left( \int_0^x \|f^*(|y-t|)\chi_{(0,x)}(y)\|^r dt\right)^{1/r}.$$

Now consider the function

$$g_{t,x}(y) = f^*(|y-t|)\chi_{(0,x)}(y), \quad 0 < t < x.$$

If $x \leq 2t$, then $g^*_{t,x}(s) = f^*(s/2)\chi_{(0,2x-2t)}(s) + f^*(s-x+t)\chi_{[2x-2t,x)}(s)$, and so

$$\begin{aligned}\|g_{t,x}\|^p &\geq \int_0^\infty g^{*p}_{t,x}(s)w(s)ds \\ &= \int_0^{2x-2t} f^{*p}(s/2)w(s)ds + \int_{2x-2t}^x f^{*p}(s-x+t)w(s)ds \\ &\geq \int_0^{2x-2t} f^{*p}(s)w(s)ds + \int_{2x-2t}^x f^{*p}(s)w(s)ds = \int_0^x f^{*p}(s)w(s)ds.\end{aligned}$$

If $x > 2t$, then $g^*_{t,x}(s) = f^*(s/2)\chi_{(0,2t)}(s) + f^*(s-t)\chi_{[2t,x)}(s)$ and

$$\|g_{t,x}\|^p \geq \int_0^{2t} f^{*p}(s/2)w(s)ds + \int_{2t}^x f^{*p}(s-t)w(s)ds \geq \int_0^x f^{*p}(s)w(s)ds.$$

Thus

$$\left(2\int_0^x f^{*r}(s)ds\right)^{1/r} \varphi(x) \geq C\left( \int_0^x \left( \int_0^x f^{*r}(s)w(s)ds\right)^{r/p} dt\right)^{1/r},$$

which yields

$$\frac{\int_0^\infty f^*(s)w(s)\chi_{(0,x)}(s)ds}{(\int_0^\infty f^*(s)^{r/p}\chi_{(0,x)}(s)ds)^{p/r}} \leq (2/C^r)^{p/r} x^{-p/r}\varphi^p(x).$$

By Sawyer's duality formula ([35], Theorem 1)

$$\sup_{0\leq f\downarrow} \frac{\int_0^\infty fg}{(\int_0^\infty f^{1/q}h)^q} \approx \left( \int_0^\infty \left(\int_0^t g\right)^{q/(1-q)} \left(\int_0^t h\right)^{-q/(1-q)} g(t)dt\right)^{1-q}$$

applied for $g(s) = w(s)\chi_{(0,x)}(s)$, $h(s) = \chi_{(0,x)}(s)$, $t < x$ and $q = p/r < 1$, it follows that

$$\left(\int_0^x t^{-q/(1-q)} W(t)^{q/(1-q)} w(t)dt\right)^{1-q} \leq Cx^{-p/r}\left(\int_0^x w(t)dt + x^p \int_x^\infty t^{-p}w(t)dt\right).$$

Now by the assumption of condition $RB_p$ on $w$,

$$\left(\int_0^x t^{-q/(1-q)} W(t)^{q/(1-q)} w(t)dt\right)^{1-q} \leq Dx^{-p/r} W_p(x).$$

Finally, applying Lemma 1 from [22], we obtain

$$\int_0^x t^{-p/r} w(t)dt \leq Ax^{-p/r} W_p(x)$$

and (v) holds. ∎

COROLLARY 3.8: *Let $1 \leq p < \infty$ and $w$ be a weight function satisfying $\int_0^1 t^{-p} w(t)dt = \int_1^\infty w = \infty$. Then*

(1) *$\Gamma_{p,w}$ has finite cotype if and only if its fundamental function $\varphi$ is regular, that is $\alpha(\varphi) > 0$.*
(2) *$\Gamma_{p,w}$ has non-trivial type if and only if $1 < p < \infty$ and $0 < \alpha(\varphi) \leq \beta(\varphi) < 1$.*

*Proof:* (1) It is well known that if $\Gamma_{p,w}$ has finite cotype then it cannot uniformly contain copies of $\ell_n^\infty$, and so by Proposition 1.13, $\alpha(\varphi) > 0$. Conversely, if we assume that $\alpha(\varphi) > 0$ and $p > 1$, then by Proposition 1.10 there exists a weight function $w_0$ such that it satisfies condition $RB_p$ and $\Gamma_{p,w} = \Gamma_{p,w_0}$, that is $\varphi$ is equivalent to $\varphi_{\Gamma_{p,w_0}}$. Hence $\alpha(\varphi_{\Gamma_{p,w_0}}) = \alpha(\varphi) > 0$ and thus there exists $r > p$ such that $\alpha(\varphi_{\Gamma_{p,w_0}}) > 1/r$. Now Theorem 3.4 gives that $\Gamma_{p,w_0}$ has non-trivial cotype and so $\Gamma_{p,w}$. If $p = 1$ then we proceed analogously applying Proposition 1.10 and Theorem 3.7.

(2) If $0 < \alpha(\varphi) \leq \beta(\varphi) < 1$ and $p > 1$, then there exists $r > 1$ such that $\beta(\varphi_{\Gamma_{p,w_0}}) = \beta(\varphi) < 1/r$. Now applying Theorem 3.4, we obtain that $\Gamma_{p,w_0}$, and hence $\Gamma_{p,w}$ has non-trivial type. Conversely, if $\Gamma_{p,w}$ has non-trivial type then its cotype must be finite, and so $\alpha(\varphi) > 0$ by (1). By Theorem 3.5, $p$ must satisfy $p > 1$. This implies that $\Gamma_{p,w_0} = \Gamma_{p,w}$ is $r$-convex for some $r > 1$, and thus by Theorem 3.4, $\beta(\varphi) = \beta(\varphi_{\Gamma_{p,w_0}}) < 1/r < 1$. ∎

The next theorem, a characterization of type and cotype of $\Gamma_{p,w}$, in view of the well known relations to convexity and concavity in Banach lattices (cf. Theorems 1.f.16, 1.f.17, and the diagrams on page 100 in [28]), is now a corollary of Propositions 1.10, 1.13 and the previous results in section 3.

THEOREM 3.9: *Let $1 \leq p < \infty$ and $\int_0^1 t^{-p} w(t)dt = \int_1^\infty w = \infty$.*

I. *$\Gamma_{p,w}$ has cotype $2 < r < \infty$ (resp. type $1 < r < 2$) if and only if $r \geq p$ and $\varphi(x)/x^{1/r}$ is pseudo-increasing (resp. $1 < r \leq p, \alpha(\varphi) > 0$ and $\varphi(x)/x^{1/r}$ is pseudo-decreasing).*

II. (a) *Let* $p \neq 2$. *Then* $\Gamma_{p,w}$ *has cotype* 2 *(resp. type* 2*) if and only if* $1 \leq p < 2$ *and* $\alpha(\varphi) > 1/2$ *(resp.* $2 < p < \infty$ *and* $0 < \alpha(\varphi) \leq \beta(\varphi) < 1/2$*).*
(b) *If* $p = 2$, *then* $\Gamma_{p,w} = \Gamma_{2,w}$ *has cotype* 2 *(resp. type* 2*) if and only if* $\varphi(x)/\sqrt{x}$ *is pseudo-increasing (resp.* $\varphi(x)/\sqrt{x}$ *is pseudo-decreasing and* $\alpha(\varphi) > 0$*).*

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